The upper forcing edge-to-vertex geodetic number of a graph

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Abstract

For a connected graph $G = (V, E)$, a set $S \subseteq E$ is called an edge-to-vertex geodetic set of $G$ if every vertex of $G$ is either incident with an edge of $S$ or lies on a geodesic joining some pair of edges of $S$. The minimum cardinality of an edge-to-vertex geodetic set of $G$ is $g_{ev}(G)$. Any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is called an edge-to-vertex geodetic basis of $G$. A subset $T \subseteq S$ is called a forcing subset for $S$ if $S$ is the unique minimum edge-to-vertex geodetic set containing $T$. A forcing subset for $S$ of minimum cardinality is a minimum forcing subset of $S$. The forcing edge-to-vertex geodetic number of $S$, denoted by $f_{ev}(S)$, is the cardinality of a minimum forcing subset of $S$. The upper forcing edge-to-vertex geodetic number of $G$, denoted by $f_{ev}^+(G)$, is $f_{ev}^+(G) = \max \{ f_{ev}(S) \}$, where the maximum is taken over all minimum edge-to-vertex geodetic sets $S$ in $G$. It is shown that the upper forcing edge-to-vertex geodetic number lies between 0 and $g_{ev}(G)$. Also, the upper forcing edge-to-vertex geodetic number of certain classes of graphs such as cycle, tree, complete graph and complete bipartite graph are determined.

**Keywords:** edge-to-vertex geodetic number, forcing edge-to-vertex geodetic number, upper forcing edge-to-vertex geodetic number.

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1 Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. For basic definitions and terminology we refer to [1]. For vertices $u$ and $v$ in a connected graph $G$, the distance $d(u, v)$ is the length of a shortest $u - v$ path in $G$. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. A geodetic set of $G$ is a set $S \subseteq V(G)$ such that every vertex of $G$ is contained in a geodesic joining some pair of vertices of $G$. The geodetic number $g(G)$ of $G$ is the minimum order of a geodetic set and any geodetic set of order $g(G)$ is called a geodetic basis of $G$. The geodetic number of a graph was introduced in [1] and further studied in [5]. A set $S \subseteq E(G)$ is called an edge-to-vertex geodetic set of $G$ if every vertex of $G$ is either incident with an edge of $S$ or lies on a geodesic joining a pair of edges of $S$. The minimum cardinality of an edge-to-vertex geodetic set of $G$ is $g_{ev}(G)$. Any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is called an edge-to-vertex geodetic basis of $G$ or a $g_{ev}$-set of $G$. The edge-to-vertex geodetic number of a graph was introduced in [12] and further studied in [7]. A vertex $v$ is an extreme vertex of a graph.
if the subgraph induced by its neighbors is complete. An edge of a connected graph $G$ is called an extreme edge of $G$ if one of its ends is an extreme vertex of $G$. For any edge $e$ in a connected graph $G$, the edge-to-edge eccentricity $e_3(e)$ of $e$ is $e_3(e) = \max \{ d(e, f) : f \in E(G) \}$. Any edge $e$ for which $e_3(e)$ is minimum is called an edge-to-edge central edge of $G$ and the set of all edge-to-edge central edges of $G$ is the edge-to-edge center of $G$. The minimum eccentricity among the edges of $G$ is the edge-to-edge radius, $\text{rad } G$, and the maximum eccentricity among the edges of $G$ is the edge-to-edge diameter, $\text{diam } G$ of $G$. Two edges $e$ and $f$ are antipodal if $d(e, f) = \text{diam } G$ or $d(G)$. This concept was studied in [10]. The forcing concept was first introduced and studied in minimum dominating sets in [2] and the same in geodetic number was introduced and studied by Chartrand and Zhang in [3]. Then the forcing concept is applied in various graph parameters viz. hull sets, matching’s, edge coverings and Steiner sets in [4, 6, 9, 8, 11] by several authors. In this paper we study the upper forcing concept in minimum edge-to-vertex geodetic set of a connected graph.

Throughout the paper $G$ denotes a connected graph with at least three vertices. The following theorems are used in the sequel.

**Theorem 1.1** (12). Let $G$ be a connected graph with size $q$. Then every end-edge of $G$ belongs to every edge-to-vertex geodetic set of $G$.

**Theorem 1.2** (12). For the complete bipartite graph $G = K_{n,n}$ ($n \geq 2$), a set $S$ of edges of $G$ is a minimum edge-to-vertex geodetic set if and only if $S$ consists of $n$ independent edges of $G$.

**Theorem 1.3** (12). For the complete bipartite graph $G = K_{m,n}$ ($2 \leq m < n$), a set $S$ of edges of $G$ is a minimum edge-to-vertex geodetic set if and only if $S$ consists of $m - 1$ independent edges of $G$ and $n - m + 1$ adjacent edges of $G$.

**Theorem 1.4** (12). For the complete graph $G = K_p$ ($p \geq 4$) with $p$ even, a set $S$ of edges of $G$ is a minimum edge-to-vertex geodetic set of $G$ if and only if $S$ consists of $\frac{p}{2}$ independent edges.

**Theorem 1.5** (12). For the complete graph $G = K_p$ ($p \geq 5$) with $p$ odd, a set $S$ of edges of $G$ is a minimum edge-to-vertex geodetic set of $G$ if and only if $S$ consists of $\frac{p-3}{2}$ independent edges and two adjacent edges of $G$.

## 2 The Forcing Edge-to-vertex Geodetic Number of a Graph

For each minimum edge-to-vertex geodetic set $S$ in a connected graph $G$, there is always some subset $T$ of $S$ such that $S$ is the unique minimum edge-to-vertex geodetic set containing $T$. The maximum of such subsets $T$ of $S$ is considered in this section.

**Definition 2.1.** Let $G$ be a connected graph and $S$ an edge-to-vertex geodetic set of $G$. A subset $T \subseteq S$ is called a forcing subset for $S$ if $S$ is the unique minimum edge-to-vertex geodetic set containing $T$. A forcing subset for $S$ of minimum cardinality is a minimum forcing subset of $S$. The
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forcing edge-to-vertex geodetic number of \( S \), denoted by \( f_{ev}(S) \), is the cardinality of a minimum forcing subset of \( S \). The upper forcing edge-to-vertex geodetic number of \( G \), denoted by \( f^+_{ev}(G) \), is

\[ f^+_{ev}(G) = \max \{ f_{ev}(S) \} \]

where the maximum is taken over all minimum edge-to-vertex geodetic sets \( S \) in \( G \).

Example 2.2. For the graph \( G \) given in Figure 1, \( S = \{ v_1v_2, v_5v_6 \} \) is the unique minimum edge-to-vertex geodetic set of \( G \) so that \( f_{ev}(S) = 0 \). For the graph \( G \) given in Figure 2, \( S_1 = \{ v_1v_2, v_3v_4, v_3v_5 \} \), \( S_2 = \{ v_1v_2, v_3v_4, v_4v_5 \} \) and \( S_3 = \{ v_1v_2, v_3v_5, v_4v_5 \} \), \( S_4 = \{ v_1v_2, v_3v_4, v_2v_5 \} \), \( S_5 = \{ v_1v_2, v_2v_3, v_4v_5 \} \) and \( S_6 = \{ v_1v_2, v_3v_5, v_2v_4 \} \) are the only \( g_{ev} \)-sets of \( G \), such that \( f_{ev}(S_1) = f_{ev}(S_2) = f_{ev}(S_3) = 2 \), and \( f_{ev}(S_4) = f_{ev}(S_5) = f_{ev}(S_6) = 1 \) so that \( f^+_{ev}(G) = \max \{ f_{ev}(S) \} = \max \{ 2, 2, 2, 1, 1 \} = 2 \).

The next theorem follows immediately from the definition of the edge-to-vertex geodetic number and the upper forcing minimum edge-to-vertex geodetic number of a connected graph \( G \).

Theorem 2.3. For every connected graph \( G \), \( 0 \leq f^+_{ev}(G) \leq g_{ev}(G) \).
Since every connected graph $G$ has one or more minimum edge-to-vertex geodetic sets and every minimum edge-to-vertex geodetic set contains at least two edges, it follows that $f_{ev}^+(G) \geq 0$. Let $S$ be a minimum edge-to-vertex geodetic set of $G$ and $T$ a forcing subset of $S$. By definition, $T \subseteq S$. This implies that, the cardinality of $T$ is less than or equal to the cardinality of $S$. That is $f_{ev}^+(G) \leq g_{ev}(G)$. 

**Proof:** Since every connected graph $G$ has one or more minimum edge-to-vertex geodetic sets and every minimum edge-to-vertex geodetic set contains at least two edges, it follows that $f_{ev}^+(G) \geq 0$. Let $S$ be a minimum edge-to-vertex geodetic set of $G$ and $T$ a forcing subset of $S$. By definition, $T \subseteq S$. This implies that, the cardinality of $T$ is less than or equal to the cardinality of $S$. That is $f_{ev}^+(G) \leq g_{ev}(G)$. 

**Remark 2.4.** The bounds in Theorem 2.3 are sharp. For the graph $G$ given in Figure 1, $f_{ev}^+(G) = 0$ and for the graph $G = K_3$, $f_{ev}^+(G) = g_{ev}(G) = 2$. Also, all the inequalities in the theorem are strict. For the graph $G$ given in Figure 2, $f_{ev}^+(G) = 2$ and $g_{ev}(G) = 3$ so that $0 < f_{ev}^+(G) < g_{ev}(G)$.

In the following, we characterize graphs $G$ for which bounds in Theorem 2.3 attained and also graph for which $f_{ev}^+(G) = 1$.

**Theorem 2.5.** Let $G$ be a connected graph. Then
(a) $f_{ev}^+(G) = 0$ if and only if $G$ has a unique minimum edge-to-vertex geodetic set.
(b) $f_{ev}^+(G) = 1$ if and only if $G$ has at least two minimum edge-to-vertex geodetic sets, in which one element of each minimum edge-to-vertex geodetic set of $G$ does not belong to any other minimum edge-to-vertex geodetic set of $G$, and
(c) $f_{ev}^+(G) = g_{ev}(G)$ if and only if there exists a minimum edge-to-vertex geodetic set of $G$ which does not contain any proper forcing subsets.

**Proof:** (a) Let $f_{ev}^+(G) = 0$. Then, by definition, $f_{ev}(S) = 0$ for some minimum edge-to-vertex geodetic set $S$ of $G$ so that the empty set $\phi$ is the minimum forcing subset for $S$. Since the empty set $\phi$ is a subset of every set, it follows that $S$ is the unique minimum edge-to-vertex geodetic set of $G$. Conversely, let $S$ be the unique minimum edge-to-vertex geodetic set of $G$. It is clear that $f_{ev}(S) = 0$ and hence $f_{ev}^+(G) = 0$.

(b) Let $f_{ev}^+(G) = 1$. Then by Theorem 2.5(a), $G$ has at least two minimum edge-to-vertex geodetic sets. Also, since $f_{ev}^+(G) = 1$, then by definition $f_{ev}(S) = 1$ for all $S$. Therefore there is a singleton subset $T$ of a minimum edge-to-vertex geodetic set $S$ of $G$ such that $T$ is not a subset of any other minimum edge-to-vertex geodetic sets of $G$. Thus one element of each $S$ does not belong to any other minimum edge-to-vertex geodetic set of $G$. Conversely, suppose that $G$ has at least two minimum edge-to-vertex geodetic sets, in which one element of each minimum edge-to-vertex geodetic set not containing any other minimum edge-to-vertex geodetic sets. It is clear that $f_{ev}(S) = 1$ for all minimum edge-to-vertex geodetic set $S$ in $G$. Hence $f_{ev}^+(G) = \max\{f_{ev}(S)\} = 1$.

(c) Let $f_{ev}^+(G) = g_{ev}(G)$. Then $f_{ev}(S) = g_{ev}(G)$ for some minimum edge-to-vertex geodetic set $S$ in $G$. Since, $q \geq 2$, $g_{ev}(G) \geq 2$ and hence $f_{ev}(S) \geq 2$. Then by Theorem 2.5(a), $G$ has at least two minimum edge-to-vertex geodetic sets and so the empty set $\phi$ is not a forcing subset for any minimum edge-to-vertex geodetic set of $G$. Since $f_{ev}(S) = g_{ev}(G)$ for some $S$, there exists some minimum edge-to-vertex geodetic sets $S$ such that no proper subset of $S$ is a forcing subset of $S$. Thus there
exists at least one minimum edge-to-vertex geodetic set of $G$ which does not contain any proper forcing subsets. Conversely, the data implies that $G$ contains more than one minimum edge-to-vertex geodetic sets such that at least one minimum edge-to-vertex geodetic set $S$ other than $S$ is a forcing subset for $S$. Hence it follows that $f^{+}_{ev}(G) = g_{ev}(G)$.

**Definition 2.6.** An edge $e$ of a connected graph $G$ is an edge-to-vertex geodetic edge of $G$ if $e$ belongs to every edge-to-vertex geodetic basis of $G$. If $G$ has a unique edge-to-vertex geodetic basis $S$, then every edge of $S$ is an edge-to-vertex geodetic edge of $G$.

**Example 2.7.** For the graph $G$ given in Figure 1, $S = \{v_1v_2, v_5v_6\}$ is the unique minimum edge-to-vertex geodetic set of $G$ so that both the edges in $S$ are edge-to-vertex geodetic edges of $G$.

**Remark 2.8.** By Theorem 1.1, each end edge of $G$ is an edge-to-vertex geodetic edge of $G$. In fact there are certain edge-to-vertex geodetic edges, which are not end edges as shown in the following example.

**Example 2.9.** For the graph $G$ given in Figure 3, $S_1 = \{v_1v_2, v_5v_6, v_7v_8\}$ and $S_2 = \{v_1v_2, v_5v_6, v_7v_8\}$ are the only $g_{ev}$-sets of $G$ so that every $g_{ev}$-set contains the edge $v_1v_2$. Hence the edge $v_1v_2$ is the unique edge-to-vertex geodetic edge of $G$, which is not an end edge of $G$.

**Theorem 2.10.** Let $G$ be a connected graph and $S$ a minimum edge-to-vertex geodetic set of $G$. Then no edge-to-vertex geodetic edge of $G$ belongs to any minimum forcing set of $S$.

**Proof:** Let $S$ be a minimum edge-to-vertex geodetic set of $G$. Let $T$ be a unique minimum forcing subset of $S$. Let $e$ be an edge-to-vertex geodetic edge of $G$. By the definition $e \in S$ for all $S$. We show that $e \notin T$ for all $T$ contained in $S$. Suppose $e$ is in any forcing subset $T$ of $S$, then $e$ does not belong to any other minimum edge-to-vertex geodetic set of $G$. This implies that $e$ is not an edge-to-vertex geodetic edge of $G$. Thus $e \notin T$ for all $T \subset S$.

**Theorem 2.11.** Let $G$ be a connected graph and $W$ be the set of all edge-to-vertex geodetic edges of $G$. Then $f^{+}_{ev}(G) \leq g_{ev}(G) - |W|$.

**Proof:** Let $S$ be a minimum edge-to-vertex geodetic set of $G$. Then $g_{ev}(G) = |S|$, $W \subseteq S$ and $S$ is the unique minimum edge-to-vertex geodetic set containing $S - W$. Thus $f^{+}_{ev}(G) \leq |S - W| \leq |S| - |W| = g_{ev}(G) - |W|$.
Corollary 2.12. If \( G \) is a connected graph with \( k \) end edges, then \( f_{ev}^+(G) \leq g_{ev}(G) - k \).

Proof: This follows from Theorems 1.1 and 2.11.

Remark 2.13. The bound in Theorem 2.11 is sharp. For the graph \( G \) given in Figure 3, \( S_1 = \{v_1v_2, v_6v_7, v_7v_8\}, S_2 = \{v_1v_2, v_5v_6, v_7v_8\} \) and \( S_3 = \{v_1v_2, v_5v_8, v_7v_7\} \) are the only \( g_{ev} \)-sets of \( G \) such that \( f_{ev}(S_1) = 2 \) and \( f_{ev}(S_2) = f_{ev}(S_3) = 1 \) so that \( f_{ev}^+(G) = \max\{f_{ev}(S)\} = 2 \) and \( g_{ev}(G) = 3 \). Also, every \( g_{ev} \)-set contains the edge \( v_1v_2 \) so that \( |W| = 1 \) hence \( f_{ev}^+(G) = g_{ev}(G) - |W| \). Also, the inequality in Theorem 2.11 can be strict. For the graph \( G \) given in Figure 4, \( S_1 = \{v_1v_2, v_3v_4, v_5v_6\}, S_2 = \{v_1v_4, v_2v_3, v_3v_6\} \) are the only two \( g_{ev} \)-sets of \( G \) such that \( f_{ev}(S_1) = f_{ev}(S_2) = 1 \) so that \( f_{ev}^+(G) = 1 \). Also \( g_{ev}(G) = 3 \). Here, \( v_5v_6 \) is the only edge-to-vertex geodetic edge of \( G \) and so \( f_{ev}^+(G) < g_{ev}(G) - |W| \).

![Figure 4](image_url)

In the following we determine the upper forcing edge-to-vertex geodetic number of some standard graphs.

Theorem 2.14. For an even cycle \( C_p(p \geq 4) \), a set \( S \subseteq E(G) \) is a minimum edge-to-vertex geodetic set if and only if \( S \) consists of antipodal edges.

Proof: Let \( p = 2k \) and let \( C_p : v_1, v_2, v_3, ..., v_k, v_{k+1}, ..., v_{2k}, v_1 \) be the cycle. Then the edges \( v_1v_2 \) and \( v_{k+1}v_{k+2} \) are antipodal edges. Let \( S = \{v_1v_2, v_{k+1}v_{k+2}\} \). Clearly, \( S \) is a minimum edge-to-vertex geodetic set of \( C_p \). Conversely, let \( S \) be a minimum edge-to-vertex geodetic set of \( C_p \). Then \( g_{ev}(C_p) = |S| \). Let \( S' \) be any set of pair of antipodal edges of \( C_p \). Then as in the first part of this theorem, \( S' \) is a minimum edge-to-vertex geodetic set of \( C_p \). Hence \( |S'| = |S| \). Thus \( S = \{uv, xy\} \). If \( uv \) and \( xy \) are not antipodal, then any vertex that is not on the \( uv - xy \) geodesic does not lie on the \( uv - xy \) geodesic. Thus \( S \) is not a minimum edge-to-vertex geodetic set, which is a contradiction.

Theorem 2.15. For an even cycle \( C_p(p \geq 4) \), \( f_{ev}^+(C_p) = 1 \).
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**Proof:** If \( p \) is even, then by Theorem 2.14, every minimum edge-to-vertex geodetic set of \( C_p \) consists of pair of antipodal edges. Hence \( C_p \) has \( p/2 \) independent minimum edge-to-vertex geodetic sets and it is clear that each singleton set is the minimum forcing set for exactly one minimum edge-to-vertex geodetic set of \( C_p \). Hence it follows from Theorem 2.5 (a) and (b) that \( f_{ev}^+(C_p) = 1 \).

**Theorem 2.16.** For an odd cycle \( C_p (p > 5) \), \( f_{ev}^+(C_p) = 3 \).

**Proof:** Let \( p \) be odd. Let \( p = 2n + 1, n = 2, 3, \ldots \). Let the cycle be \( C_p : v_1, v_2, v_3, \ldots, v_{2n+1}, v_1 \).

If \( S = \{uv, xy\} \) is any set of two edges of \( C_p \), then no edge of the \( uv - xy \) longest path lies on the \( uv - xy \) geodesic in \( C_p \) and so no two element subset of \( C_p \) is an edge-to-vertex geodetic set of \( C_p \). Now, it is clear that the sets \( S_1 = \{v_1v_2, v_2v_3v_n+2, v_2v_{2n+1}\}, S_2 = \{v_1v_2, v_nv_1+1, v_2v_{2n+1}\}, S_3 = \{v_2v_3, v_{n+2}v_{n+3}, v_{2n+1}\}, \ldots, S_{2n} = \{v_nv_1+1, v_2v_{2n+1}v_{n-1}v_n\}, S_{2n+1} = \{v_nv_1+1, v_2v_{2n+1}, v_{n-1}v_n\} \) are the minimum edge-to-vertex geodetic sets of \( C_p \). (Note that there are more minimum edge-to-vertex geodetic sets of \( C_p \), for example \( S = \{v_n+2v_{n+3}, v_1v_2v_{2n+1}\} \) is a minimum edge-to-vertex geodetic set different from these). It is clear from the minimum edge-to-vertex geodetic sets \( S_i (1 \leq i \leq 2n + 1) \) that each \( \{v_iv_{i+1}\} (1 \leq i \leq 2n) \) and \( \{v_{2n+1}v_1\} \) is a subset of more than one minimum edge-to-vertex geodetic set \( S_i (1 \leq i \leq 2n + 1) \). Hence it follows from Theorem 2.5 (b) and (c) that \( f_{ev}^+(C_p) \leq 3 \). Since \( S_2 \) is the unique minimum edge-to-vertex geodetic set containing \( T = \{v_1v_2, v_{2n+1}v_1\} \), it follows that \( f_{ev}^+(S_2) = 2 \). But it is easily verified that the two element subsets of \( S_1 \) are contained in more than one minimum edge-to-vertex geodetic set \( S_i (1 \leq i \leq 2n + 1) \) so that \( f_{ev}^+(S_1) \neq 2 \) and hence \( f_{ev}^+(S_1) = 3 \). Thus \( f_{ev}^+(C_p) = 3 \).

**Theorem 2.17.** For the complete bipartite graph \( G = K_{n,n} (n \geq 2) \), \( f_{ev}^+(G) = n - 1 \).

**Proof:** Let \( X = \{u_1, u_2, \ldots, u_n\} \) and \( Y = \{v_1, v_2, \ldots, v_m\} \) be a partition of \( G \). Let \( S \) be a minimum edge-to-vertex geodetic set of \( G \). Then by Theorem 1.2, every element of \( S \) is independent and \( |S| = n \). We show that \( f_{ev}^+(G) = n - 1 \).

**Case(i):** Suppose that \( f_{ev}^+(G) \leq n - 2 \). Then there exists a forcing subset \( T \) of \( S \) such that \( S \) is the unique minimum edge-to-vertex geodetic set of \( G \) containing \( T \) and \( |T| \leq n - 2 \). Hence there exists at least two edges \( u_iv_j, u_{i}v_{m} \in S \) such that \( u_iv_j, u_{i}v_{m} \notin T \) and \( i \neq l, j \neq m \). Then \( S_1 = S - \{u_iv_j, u_{i}v_{m}\} \cup \{u_{i}v_{m}, u_{i}v_j\} \) is a set of \( n \) independent edges of \( G \). By Theorem 1.2, \( S_1 \) is a minimum edge-to-vertex geodetic set of \( G \) which is a contradiction to \( T \) is a forcing subset of \( S \). Hence \( f_{ev}^+(G) \leq n - 2 \) is not possible.

**Case(ii):** Suppose that \( f_{ev}^+(G) > n - 1 \). By Theorem 2.5(c), \( f_{ev}^+(G) = n \). Then there exists a forcing subset \( T \) of \( S \) such that \( S \) is the unique minimum edge-to-vertex geodetic set of \( G \) containing \( T \) and \( |T| = n \). Hence all the proper subsets of \( S \) having a single element, two elements, three elements, \ldots, \( n - 1 \) elements are contained in more than one minimum edge-to-vertex geodetic sets of \( G \). Let \( F \) be a proper subset of \( S \) with cardinality \( n - 1 \). Let \( S_1 \) and \( S_2 \) be the two minimum edge-to-vertex geodetic sets of \( G \) containing \( F \). Since \( S_1 \) and \( S_2 \) have \( n - 1 \) elements as common, the other \( nth \) element of \( S_1 \)
and $S_2$ is also same. Thus we get more than one minimum edge-to-vertex geodetic set with the same $n$ independent edges, which is a contradiction to $T$ is a forcing subset of $S$. Hence $f_{ev}^+(G) = n$ is not possible. Thus $f_{ev}^+(G) = n - 1$.

**Theorem 2.18.** For the complete bipartite graph $G = K_{m,n}(2 \leq m < n)$, $f_{ev}^+(G) = n - 1$.

**Proof:** Let $X = \{u_1, u_2, \ldots, u_n\}$ and $Y = \{v_1, v_2, \ldots, v_m\}$ be a partition of $G$. Let $S$ be a minimum edge-to-vertex geodetic set of $G$. Then by Theorem 1.3, $S = S_1 \cup S_2$, where $S_1$ consists of $m - 1$ independent edges and $S_2$ consists of $n - m + 1$ adjacent edges and $|S| = n$. We show that $f_{ev}^+(G) = n - 1$.

**Case(i):** Suppose that $f_{ev}^+(G) \leq n - 2$. Then there exists a forcing subset $T$ of $S$ such that $T$ is the unique minimum edge-to-vertex geodetic set of $G$ containing $T$ and $|T| \leq n - 2$. Hence there exists at least two edges $x, y \in S$ such that $x, y \notin T$. Let us assume that $S_2 = \{u_kv_{l1}, u_kv_{l2}, \ldots, u_kv_{ln-m+1}\}$. Suppose that $x, y \in S_1$. Then $x = u_iv_j$ and $y = u_lv_m$ such that $i \neq l$ and $j \neq m$. Now, $S_3 = S - \{x, y\} \cup \{u_iv_m, u_lv_j\}$ consists of $m - 1$ independent edges and $n - m + 1$ adjacent edges of $G$ and also containing $T$. By Theorem 1.3, $S_3$ is a minimum edge-to-vertex geodetic set of $G$, which is a contradiction to $T$ is a forcing subset of $G$. Suppose that $x, y \in S_2$. Let $x = u_kv_{l1}$ and $y = u_kv_{l2}$. Let $u_iv_j$ be an edge of $S_1$. Now, join the vertices $v_{l2}, v_{l3}, \ldots, v_{ln-m+1}$ to $u_i$. Now $S_4 = S_1 - \{u_iv_j\} \cup \{u_kv_{l1}\} \cup \{u_iv_{l1}, u_iv_{l2}, u_iv_{l3}, \ldots, u_iv_{ln-m+1}\}$ consists of $m - 1$ independent edges and $n - m + 1$ adjacent edges of $G$. By Theorem 1.3, $S_4$ is a minimum edge-to-vertex geodetic set of $G$ containing $T$, which is a contradiction. Suppose that $x \in S_1$ and $y \in S_2$. Let $x = u_iv_j$ and $y = u_kv_{l1}$. $S_5 = S_1 - \{u_iv_j\} \cup \{u_iv_{l1}\} \cup \{u_kv_{j}, u_kv_{l2}, u_kv_{l3}, \ldots, u_kv_{ln-m+1}\}$ consists of $m - 1$ independent edges and $n - m + 1$ adjacent edges of $G$ and also containing $T$. By Theorem 1.3, $S_5$ is a minimum edge-to-vertex geodetic set of $G$, which is a contradiction to that $T$ is a forcing subset of $G$. Hence $f_{ev}^+(G) \leq n - 2$ is not possible.

**Case(ii):** Suppose that $f_{ev}^+(G) > n - 1$. This implies that, by Theorem 2.5(c), $f_{ev}^+(G) = n$. Then there exists a forcing subset $T$ of $S$ such that $S$ is the unique minimum edge-to-vertex geodetic set of $G$ containing $T$ and $|T| = n$. Hence all the proper subsets of $S$ containing a single element, two elements, three elements, ..., $n - 1$ elements are contained in more than one minimum edge-to-vertex geodetic sets of $G$. Consider a proper subset $F$ of cardinality $n - 1( m - 2$ independent edges and $n - m + 1$ adjacent edges). Since $f_{ev}^+(G) = n$, it is clear that the proper subset $F$ lies more than one minimum edge-to-vertex geodetic sets of $G$, say $S_1$ and $S_2$. Now $S_1$ and $S_2$ have $n - 1$ elements in common. This implies that the other $n^{th}$ independent edge of $S_1$ and $S_2$ is also same. Thus we get more than one minimum edge-to-vertex geodetic set of $G$ with the same $n$ independent edges which is a contradiction to that $T$ is a forcing subset of $S$. Hence $f_{ev}^+(G) = n - 1$.

**Theorem 2.19.** For the complete graph $G = K_p(p \geq 4)$ with $p$ even, $f_{ev}^+(G) = \frac{p-2}{2}$.

**Proof:** The proof is similar to the proof of Theorem 2.17.

**Theorem 2.20.** For the complete graph $G = K_p(p \geq 5)$ with $p$ odd, $f_{ev}^+(G) = \frac{p-1}{2}$.
Proof: The proof is similar to the proof of Theorem 2.18.

Theorem 2.21. For a non trivial tree of size $q \geq 2$, $f_{ev}^{+}(G) = 0$.

Proof: Let $G$ be a tree of size $q$. Then by Theorem 1.1, every pendent edge of $G$ belongs to every edge-to-vertex geodetic set of $G$. But it is clear that, in a tree, the set of all pendent edges of $G$ is the unique minimum edge-to-vertex geodetic set of $G$. Now, it follows from Theorem 2.5(a) that $f_{ev}^{+}(G) = 0$.

Theorem 2.22. For a star $G = K_{1,q}$, $f_{ev}^{+}(G) = 0$.

Proof: This follows from Theorem 2.21.

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