On edge irregularity strength of subdivision of star $S_n$

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Abstract

In this paper, we study the total edge irregularity strength of some well known graphs. An edge irregular total $k$-labeling $\varphi: V \cup E \rightarrow \{1, 2, \ldots, k\}$ of a graph $G = (V, E)$ is a labeling of vertices and edges of $G$ in such a way that for any different edges $xy$ and $x'y'$ their weights are distinct. The total edge irregularity strength $tes(G)$ is defined as the minimum $k$ for which $G$ has an edge irregular total $k$-labeling. Also, we determine the exact value of the total edge irregularity strength of subdivision of star $S_n$.

Keywords: Irregularity strength, Total edge irregularity strength, Edge irregular total labeling, Subdivision of star.

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1 Introduction

Bača, Jendroľ, Miller and Ryan [3] defined the notion of an edge irregular total $k$-labeling of a graph $G = (V, E)$ to be a labeling of the vertices and edges of $G$, $\varphi: V \cup E \rightarrow \{1, 2, \ldots, k\}$ such that, the edge weights $wt_{\varphi}(uv) = \varphi(u) + \varphi(uv) + \varphi(v)$ are different for all edges. The minimum $k$ for which the graph $G$ has an edge irregular total $k$-labeling is called the total edge irregularity strength of $G$, $tes(G)$.

The motivation for the definition of the total edge irregularity strength came from irregular assignments and the irregularity strength of graphs introduced by Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba [6]. An irregular assignment is a $k$-labeling of the edges $\phi: E \rightarrow \{1, 2, \ldots, k\}$ such that the sum of the labels of edges incident with a vertex is different for all the vertices of $G$ and the smallest $k$ for which there is an irregular assignments is the irregularity strength and is denoted by $s(G)$.

Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [4, 7, 9, 14]. Karoński, Łuczak and Thomason [12] conjectured that the edges of every connected graph of order at least 3 can be assigned labels from $\{1, 2, 3\}$ such that for all the pairs of adjacent vertices, the sums of the labels of the incident edges are different.

We mention the following result from [3] giving a lower bound on the total edge irregularity strength of a graph:

$$tes(G) \geq \max \left\{ \left\lfloor \frac{|E(G)| + 2}{3} \right\rfloor, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\},$$

where $\Delta(G)$ is the maximum degree of $G$. The exact values of the total edge irregularity strength for paths, cycles, stars, wheels and friendship graphs are determined in [3].
Recently Ivančo and Jendroľ [8] posed the following conjecture:

**Conjecture 1.1.** [8] Let $G$ be an arbitrary graph different from $K_5$. Then

$$\text{tes}(G) = \max \left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}. \quad (2)$$

Conjecture 1 has been verified for trees in [8], for complete graphs and complete bipartite graphs in [10] and [11], for the Cartesian product of two paths $P_n \square P_m$ in [13], for corona product of a path with certain graphs in [15], for large dense graphs with $\frac{|E(G)| + 2}{3} \leq \frac{\Delta(G) + 1}{2}$ in [5], for the categorical product of two paths $P_n \times P_m$ in [2] and for the categorical product of a cycle and a path $C_n \times P_m$ in [1].

Motivated by [1], [3] and [15] we investigate the total edge irregularity strength of subdivision of a star $S_n$. In [16], for $m \geq 0$ and $n \geq 3$, let $S_n^m$ be a graph obtained by inserting $m$ vertices to every edge of a star $S_n$. Thus, the star $S_n$ can be written as $S_n^0$. The graph $S_n^m$ is given in Figure 1.

![Image](Figure 1: The graph of $S_n^m$)

We define the vertex set and the edge set of the graph $S_n^m$ as follows:

$$V(S_n^m) = \{ c, x_{i,j} : i \in [1, n], j \in [1, m + 1] \},$$

and

$$E(S_n^m) = \{ cx_{i,1}, x_{i,j-1}x_{i,j} : i \in [1, n], j \in [2, m + 1] \}. $$
Clearly, a graph $S_n^m$ has $mn + n + 1$ vertices and $mn + n$ edges. Among these vertices, one vertex has degree $n$, $n$ vertices have degree one, and the remaining vertices have degree two. As the maximum degree $\Delta(S_n^m) = n$, then (1) implies that $tes(S_n^m) \geq \lceil \frac{mn+n+2}{3} \rceil$. To show that $\lceil \frac{mn+n+2}{3} \rceil$ is an upper bound for the $tes(S_n^m)$, we describe an edge irregular total $\lceil \frac{mn+n+2}{3} \rceil$-labeling for $S_n^m$.

**Theorem 1.2.** For $n \geq 3$, $tes(S_n^m) = \lceil \frac{2n+2}{3} \rceil$.

**Proof.** The inequality $tes(S_n^m) \geq \lceil \frac{2n+2}{3} \rceil$ follows from (1). To prove $tes(S_n^m) \leq \lceil \frac{2n+2}{3} \rceil$, we split the edge set of $S_n^m$ in mutually disjoint subsets:

$A_i = \{c x_{i,1}\}$ for $1 \leq i \leq n$ and $B_i = \{x_{i,1}, x_{i,2}\}$ for $1 \leq i \leq n$.

Let $k = \lceil \frac{2n+2}{3} \rceil$ and define a total $k$-labeling $\psi_i : V \cup E \rightarrow \{1, 2, \ldots, k\}$ with $\psi_i(c) = 1$ as follows:

**Case 1.** when $n \equiv 0(\text{mod}3)$

\[
\psi_1(x_{i,j}) =
\begin{cases}
1, & \text{if } 1 \leq i \leq \frac{n}{3}, j = 1 \\
1 + i - \frac{n}{3}, & \text{if } \frac{n}{3} + 1 \leq i \leq n, j = 1 \\
k - 1, & \text{if } 1 \leq i \leq n, j = 2
\end{cases}
\]

\[
\psi_1(A_i) =
\begin{cases}
i, & \text{if } 1 \leq i \leq \frac{n-3}{3} \\
\frac{n}{3}, & \text{if } \frac{n}{3} \leq i \leq n
\end{cases}
\]

\[
\psi_1(B_i) =
\begin{cases}
\frac{n+3}{3}, & \text{if } 1 \leq i \leq \frac{n}{3} \\
k - 1, & \text{if } \frac{n+3}{3} \leq i \leq n
\end{cases}
\]

Under the labeling $\psi_1$, the total weights of the edges are as follows:

(i) edges in $A_i$ receive $2 + i$ for $1 \leq i \leq n$,

(ii) edges in $B_i$ receive $\frac{3k+3i+n+3}{3}$ for $1 \leq i \leq \frac{n}{3}$ and $\frac{6k+3i-n}{3}$ for $\frac{n+3}{3} \leq i \leq n$.

**Case 2.** when $n \equiv 1(\text{mod}3)$

\[
\psi_1(x_{i,j}) =
\begin{cases}
1, & \text{if } 1 \leq i \leq \frac{n-1}{3}, j = 1 \\
2 + i - \frac{n+2}{3}, & \text{if } \frac{n+2}{3} \leq i \leq n, j = 1 \\
k - 1, & \text{if } 1 \leq i \leq n, j = 2
\end{cases}
\]

\[
\psi_1(A_i) =
\begin{cases}
i, & \text{if } 1 \leq i \leq \frac{n-1}{3} \\
\frac{n+1}{3}, & \text{if } \frac{n+2}{3} \leq i \leq n
\end{cases}
\]

\[
\psi_1(B_i) =
\begin{cases}
\frac{n+3i-1}{3}, & \text{if } 1 \leq i \leq \frac{n-1}{3} \\
k - 2, & \text{if } \frac{n+2}{3} \leq i \leq n
\end{cases}
\]

Under the labeling $\psi_1$, the total weights of the edges are as follows:

(i) edges in $A_i$ receive $2 + i$ for $1 \leq i \leq n$,

(ii) edges in $B_i$ receive $\frac{3k+2i+n+3}{3}$ for $1 \leq i \leq \frac{n-1}{3}$ and $\frac{6k+3i-n-2}{3}$ for $\frac{n+2}{3} \leq i \leq n$. 

Case 3. when \( n \equiv 2 (\text{mod} 3) \)

\[
\psi_1(x_{i,j}) = \begin{cases} 
1, & \text{if } 1 \leq i \leq \frac{n+1}{3}, j = 1 \\
2 + i - \frac{n+4}{3}, & \text{if } \frac{n+4}{3} \leq i \leq n, j = 1 \\
k, & \text{if } 1 \leq i \leq n, j = 2 
\end{cases}
\]

\[
\psi_1(A_i) = \begin{cases} 
i, & \text{if } 1 \leq i \leq \frac{n+1}{3} \\
\frac{n+1}{3}, & \text{if } \frac{n+4}{3} \leq i \leq n \\
\frac{n+3i+1}{3}, & \text{if } 1 \leq i \leq \frac{n+1}{3} \\
k, & \text{if } \frac{n+4}{3} \leq i \leq n 
\end{cases}
\]

\[
\psi_1(B_i) = \begin{cases} 
i, & \text{if } 1 \leq i \leq \frac{n+1}{3} \\
\frac{n+1}{3}, & \text{if } \frac{n+4}{3} \leq i \leq n \\
\frac{n+3i+1}{3}, & \text{if } 1 \leq i \leq \frac{n+1}{3} \\
k, & \text{if } \frac{n+4}{3} \leq i \leq n 
\end{cases}
\]

Under the labeling \( \psi_1 \), the total weights of the edges are as follows:

(i) edges in \( A_i \) receive the integer \( 2 + i \) for \( 1 \leq i \leq n \),

(ii) edges in \( B_i \) receive the integer \( \frac{3k+4+9n}{3} \) for \( 1 \leq i \leq \frac{n+1}{3} \) and \( \frac{6k+3+i-n+2}{3} \) for \( \frac{n+4}{3} \leq i \leq n \).

It can be easily verified that the vertex and edge labels are atmost \( k \) and the edge-weights are pairwise distinct. Thus, the resulting total labeling is the desired edge irregular \( k \)-labeling. This concludes the proof. ■

**Theorem 1.3.** For \( n \geq 3 \), \( tes(S_n^2) = \lceil \frac{3n+2}{3} \rceil \).

**Proof.** The inequality \( tes(S_n^2) \geq \lceil \frac{3n+2}{3} \rceil \) follows from (1). To prove the equality we split the edge set of \( S_n^2 \) in mutually disjoint subsets:

\( A_i = \{cx_{i,1}\}, B_i = \{x_{i,1}x_{i,2}\} \) and \( C_i = \{x_{i,2}x_{i,3}\} \) for \( 1 \leq i \leq n \).

Let \( k = \lceil \frac{3n+2}{3} \rceil \). Define a total \( k \)-labeling \( \psi_2 : V \cup E \to \{1, 2, \ldots, k\} \) such that \( \psi_2(c) = 1 \) and for \( 1 \leq i \leq n \),

\[
\psi_2(x_{i,j}) = \begin{cases} 
i, & \text{if } j = 1 \\
 i + 1, & \text{if } j = 2 \\
k, & \text{if } j = 3 
\end{cases}
\]

\( \psi_2(A_i) = 1, \ \psi_2(B_i) = k - i, \ \psi_2(C_i) = k - 1 \),

Under the labeling \( \psi_2 \), the total weights of the edges are as follows:

(i) edges in \( A_i \) receive \( 2 + i \) for \( 1 \leq i \leq n \),

(ii) edges in \( B_i \) receive \( k + 1 + i \) for \( 1 \leq i \leq n \),

(iii) edges in \( C_i \) receive \( 2k + i \) for \( 1 \leq i \leq n \).

It can be easily verified \( \psi_2 \) is an edge irregular total labeling having the required property. ■

**Theorem 1.4.** For \( n \geq 3 \), \( tes(S_n^3) = \lceil \frac{4n+2}{3} \rceil \).

**Proof.** The inequality \( tes(S_n^3) \geq \lceil \frac{4n+2}{3} \rceil \) from (1). To prove the equality we split the edge set of \( S_n^3 \) in mutually disjoint subsets:

\( A_i = \{cx_{i,1}\}, B_i = \{x_{i,1}x_{i,2}\}, C_i = \{x_{i,2}x_{i,3}\}, D_i = \{x_{i,3}x_{i,4}\} \) for \( 1 \leq i \leq n \).
Let $k = \lceil \frac{4n+2}{3} \rceil$. We define a total $k$-labeling $\psi_3$ such that $\psi_3(c) = 1$ and for $1 \leq i \leq n$,

$$\psi_3(x_{i,j}) = \begin{cases} i, & \text{if } j = 1 \\ i + 1, & \text{if } j = 2 \\ k, & \text{if } j = 3, 4 \end{cases}$$

$\psi_3(A_i) = 1$, $\psi_3(B_i) = n + 1 - i$,

$$\psi_3(C_i) = \begin{cases} \frac{2n}{3} & \text{when } n \equiv 0(\text{mod}3) \\ \frac{2n+1}{3} & \text{when } n \equiv 1(\text{mod}3) \\ \frac{2n-1}{3} & \text{when } n \equiv 2(\text{mod}3) \end{cases}$$

$$\psi_3(D_i) = \begin{cases} \frac{n}{3} + i & \text{when } n \equiv 0(\text{mod}3) \\ \frac{5n+4}{3} - k + i & \text{when } n \equiv 1(\text{mod}3) \\ \frac{5n+2}{3} - k + i & \text{when } n \equiv 2(\text{mod}3) \end{cases}$$

Under the labeling $\psi_3$,

(i) edges in $A_i$ receive $2 + i$ for $1 \leq i \leq n$,

(ii) edges in $B_i$ receive $n + 2 + i$ for $1 \leq i \leq n$.

**Case 1.** when $n \equiv 0(\text{mod}3)$

(i) edges in $C_i$ receive $k + 1 + \frac{2n}{3} + i$ for $1 \leq i \leq n$,

(ii) edges in $D_i$ receive $2k + 2 + i$ for $1 \leq i \leq n$.

**Case 2.** when $n \equiv 1(\text{mod}3)$

(i) edges in $C_i$ receive $k + 2 + \frac{2n-2}{3} + i$ for $1 \leq i \leq n$,

(ii) edges in $D_i$ receive $2k + \frac{n}{3} + i$ for $1 \leq i \leq n$.

**Case 3.** when $n \equiv 2(\text{mod}3)$

(i) edges in $C_i$ receive $k + 1 + \frac{2n-1}{3} + i$ for $1 \leq i \leq n$,

(ii) edges in $D_i$ receive $k + \frac{5n+4}{3} + i$ for $1 \leq i \leq n$.

It can be easily verified that the vertex and edge labels are atmost $k$ and the edge-weights are pairwise distinct. Thus, the resulting total labeling is the desired edge irregular $k$-labeling. \hfill \blacksquare

**Theorem 1.5.** For $4 \leq m \leq 5$ and $n \geq 3$, $tes(S_n^m) = \left\lceil \frac{(m+1)n+2}{3} \right\rceil$.

**Proof.** The inequality $tes(S_n^m) \geq \left\lceil \frac{(m+1)n+2}{3} \right\rceil$ follows from (1). To prove the equality we split the edge set of $S_n^m$ in mutually disjoint subsets:

$A_{i,1} = \{ex_{i,1}\}$ for $1 \leq i \leq n$

$A_{i,j} = \{x_{i,j-1}x_{i,j}\}$ for $1 \leq i \leq n$, $2 \leq j \leq m + 1$

Let $k = \left\lceil \frac{(m+1)n+2}{3} \right\rceil$. Define a total $k$-labeling $\psi_4$ such that $\psi_4(c) = 1$ and for $1 \leq i \leq n$,

$$\psi_4(x_{i,j}) = \begin{cases} i, & \text{if } j = 1 \\ i + 1, & \text{if } j = 2 \\ n, & \text{if } j = 3 \\ k, & \text{if } j = 4, 5, 6 \end{cases}$$
\[ \psi_4(A_{i,1}) = 1, \quad \psi_4(A_{i,2}) = n + 1 - i, \]
\[ \psi_4(A_{i,3}) = n + 1, \quad \psi_4(A_{i,4}) = 2n - k + 2 + i, \]
\[ \psi_4(A_{i,5}) = 4n - 2k + 2 + i, \quad \psi_4(A_{i,6}) = 5n - 2k + 2 + i, \]

Under the labeling \( \psi_4 \), the edges in \( A_{i,j} \) receive weights \((j - 1)n + 2 + i\) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m + 1 \).

It is easy to verify that \( \psi_4 \) is an edge irregular total labeling having the required property. \( \blacksquare \)

**Theorem 1.6.** For \( n \geq 3 \), \( tes(S_n^{6}) = \lceil \frac{7n + 2}{3} \rceil \).

**Proof.** We have \( tes(S_n^{6}) \geq \lceil \frac{7n + 2}{3} \rceil \) from (1). To prove the equality we split the edge set of \( S_n^{6} \) in mutually disjoint subsets:

\[ A_{i,1} = \{ cx_{i,1} \} \text{ for } 1 \leq i \leq n \]
\[ A_{i,j} = \{ x_{i,j-1}x_{i,j} \} \text{ for } 1 \leq i \leq n, \quad 2 \leq j \leq 7. \]

Let \( k = \lceil \frac{7n + 2}{3} \rceil \). Define the total \( k \)-labeling \( \psi_5 \) such that \( \psi_5(c) = 1 \) and for \( 1 \leq i \leq n \),

\[ \psi_5(x_{i,j}) = \begin{cases} 
  i, & \text{if } j = 1 \\
  i + 1, & \text{if } j = 2 \\
  n - 1 + i, & \text{if } j = 3 \\
  n + i, & \text{if } j = 4 \\
  k, & \text{if } j = 5, 6, 7 
\end{cases} \]

\[ \psi_5(A_{i,1}) = 1, \quad \psi_5(A_{i,2}) = n + 1 - i, \]
\[ \psi_5(A_{i,3}) = n + 2 - i, \quad \psi_5(A_{i,4}) = n + 3 - i, \]
\[ \psi_5(A_{i,5}) = 3n - k + 2, \quad \psi_5(A_{i,6}) = 5n - 2k + 2 + i, \quad \psi_5(A_{i,7}) = 6n - 2k + 2 + i, \]

Under the labeling \( \psi_5 \), the edges in \( A_{i,j} \) receive weights \((j - 1)n + 2 + i\) for \( 1 \leq i \leq n \), \( 1 \leq j \leq 7 \).

It can be easily verified that \( \psi_5 \) is an edge irregular total labeling having the required property. \( \blacksquare \)

**Theorem 1.7.** For \( 7 \leq m \leq 8 \) and \( n \geq 3 \), \( tes(S_n^{m}) = \lceil \frac{(m+1)n+2}{3} \rceil \).

**Proof.** The inequality \( tes(S_n^{m}) \geq \lceil \frac{(m+1)n+2}{3} \rceil \) follows from (1). To prove the equality we split the edge set of \( S_n^{m} \) in mutually disjoint subsets:

\[ A_{i,1} = \{ cx_{i,1} \} \text{ for } 1 \leq i \leq n \]
\[ A_{i,j} = \{ x_{i,j-1}x_{i,j} \} \text{ for } 1 \leq i \leq n, \quad 2 \leq j \leq m + 1 \]

First we construct the vertex labeling \( \psi_6 \) for \( 1 \leq i \leq n \) with \( \psi_6(c) = 1 \) and \( k = \lceil \frac{(m+1)n+2}{3} \rceil \).

\[ \psi_6(x_{i,j}) = \begin{cases} 
  i, & \text{if } j = 1 \\
  i + 1, & \text{if } j = 2 \\
  n - 1 + i, & \text{if } j = 3 \\
  n + i, & \text{if } j = 4 \\
  n + 1 + i, & \text{if } j = 5 
\end{cases} \]
Case 1. when $m = 7$
\[ \psi_6(x_{i,j}) = \begin{cases} k, & \text{if } 1 \leq i \leq n, j = 6, 7, 8 \\ \end{cases} \]
Case 2. when $m = 8$
\[ \psi_6(x_{i,j}) = \begin{cases} n + 2 + i, & \text{if } 1 \leq i \leq n, j = 6 \\ k, & \text{if } 1 \leq i \leq n, j = 7, 8, 9 \\ \end{cases} \]

Now we define edge labeling $\psi_6$ as follows:

\[ \begin{align*}
\psi_6(A_{i,1}) &= 1, \\
\psi_6(A_{i,2}) &= n + 1 - i, \\
\psi_6(A_{i,3}) &= n + 2 - i, \\
\psi_6(A_{i,4}) &= n + 3 - i, \\
\psi_6(A_{i,5}) &= 2n + 1 - i, \\
\end{align*} \]

\[ \begin{align*}
\psi_6(A_{i,6}) &= 4n - k + 1, \\
\psi_6(A_{i,7}) &= 6n - 2k + 2 + i, \\
\psi_6(A_{i,8}) &= 7n - 2k + 2 + i, \\
\end{align*} \]

Under the labeling $\psi_5$, the edges in $A_{i,j}$ receive weights $(j - 1)n + 2 + i$ for $1 \leq i \leq n, 1 \leq j \leq (m + 1)$.

It can be easily verified that the vertex and edge labels are atmost $k$ and the edge-weights of the edges from the sets $A_{i,j}$ for $i \in [1, n - 1], j \in [1, m + 1]$ are pairwise distinct. Thus, the resulting total labeling is the desired edge irregular $k$-labeling. This concludes the proof.

Open Problem. We conclude the paper with the following open problem.

For $m \geq 9, n \geq 3$, determine the total edge irregular strength of subdivision of star $S_n$.

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References


